Preferential growth: Exact solution of the time-dependent distributions

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We consider a preferential growth model where particles are added one by one to the system consisting of clusters of particles. A new particle can either form a new cluster (with probability q) or join an already existing cluster with a probability proportional to the size thereof. We calculate exactly the probability $\mathcal{P}_i(k,t)$ that the size of the ith cluster at time t is k. We analyze the asymptotics, the scaling properties of the size distribution and of the mean size, as well as the relation of our system to recent network models.

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I. INTRODUCTION

Nonuniform growth is inherently present in a broad class of phenomena including the development of biological populations, communication networks, or economic systems like incomes of persons or companies [1–7]. In many cases it is obvious to assume that in a system consisting of groups or clusters of units the attachment of a new entity to one of the groups depends on the already achieved strength or size of that particular group. Simon [4] analyzed a simple model of this kind where the growth probability was proportional to the cluster size and he gave exact results for the time independent size distribution. Referring to the examples of words in a book or personal incomes Simon derived a power law distribution of cluster sizes. Recently, in the search for an explanation of the widely observed scale invariance of large networks like the World Wide Web (WWW) [1–3], the Internet or power networks [5], and scientific citation [6], the idea of preferential growth has been applied to evolving graphs [7]. It turned out that such graphs behave remarkably: They have ''small world'' properties [8] and the distribution of the strength of vertices (number of edges from or to a vertex) is scale free, provided that the probability of linking a vertex with a new one is proportional to its strength [9]. This class of models represent a new mechanism for ''self-organized criticality'' [10]. The idea of preferential growth seems to be essential in economic systems, too, where clustering of companies, e.g., according to their market behavior, follows such a pattern [11].

These models have been treated by different tools including simulations, continuum or mean field theories [12], and exact calculations [4,13] by which information has been accumulated about the asymptotic behavior and the time dependence of the global distribution functions. However, much less attention has been paid to the full time-dependent solution of the problem. The aim of the present work is to give such a solution of a particular model.

The paper is organized as follows. In Sec. II we define the model and the quantities of interest as well as we present the basic master equation. In Sec. III the main steps of the full time dependent analytic solution is given and the consequences for the steady state and the integrated distributions are drawn. Section IV contains the analysis about the asymptotics and scaling. In Sec. V we present a discussion of our results. The paper concludes with two appendices containing some details of the calculations.

II. MODEL

We model a growing system which consists of groups of different sizes. At the beginning ($t=1$) we have one group with one element in it. At each time step we add a new element to the system. With probability p it will belong to one of the existing groups. The probability that it joins the ith group is proportional to the size of the group (k_i/N); see Fig. 1. (The number of elements is equal to the time, $N=t$, because the system size is rising by one in each time step.) With probability $q=1-p$ the new element will belong to a new group.

The process can be described by the following master equation:

$$\mathcal{P}_i(k,t)=p\frac{(k-1)}{t-1}\mathcal{P}_i(k-1,t-1)+p\left(1-\frac{k}{t-1}\right)\mathcal{P}_i(k,t-1)+(1-p)\mathcal{P}_i(k,t-1)+(1-p)\Pi_{i-1}(t-1)\times\delta_{k,1}(1-\delta_{i,1}), \quad (1)$$

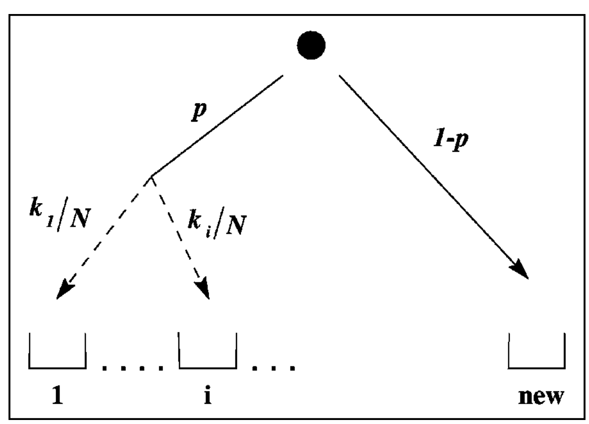

FIG. 1. Demonstration of the model. The black point on the top denotes the new incoming element, the boxes on the bottom are the groups.

where $\mathcal{P}_i(k,t)$ is the probability that at time t there are k elements in the group i, and $\Pi_i(t)$ is the probability that at time t there are i groups in the system:

$$\Pi_i(t)=\binom{t-1}{i-1}p^{t-1-(i-1)}(1-p)^{i-1}. \qquad (2)$$

In the following we introduce some important quantities and their definitions.

Given the size distribution of the individual groups, $\mathcal{P}_i(k,t)$, the size distribution of the total system can be calculated as their average:

$$\mathbf{P}(k,t)=\frac{1}{t}\sum_{i=1}^{t}\mathcal{P}_i(k,t). \qquad (3)$$

In the long time limit this quantity approximates to a stationary value: $\mathbf{P}(k)=\lim_{t\to\infty}\mathbf{P}(k,t)$. The mean of the ith group size is

$$\langle k_i\rangle(t)=\sum_{k=1}^{t-i+1}k\,\mathcal{P}_i(k,t). \qquad (4)$$

The reason that the upper limit of the above sum is not infinity is that $\mathcal{P}_i(k,t)=0$ if $k>t-i+1$.

III. ANALYTIC CALCULATIONS

A. Asymptotic distribution of group size

In the first step we calculate the group size distribution in the asymptotic case, $\mathbf{P}(k)$. The exact analytic formula for $\mathbf{P}(k)$ was already calculated in Refs. [4] and [13]; we present it here to see the dependence of the exponent on the parameter p.

If we sum up Eq. (1) for $i=1\ldots t$, we get

$$t\,\mathbf{P}(k,t)=(t-1-pk)\mathbf{P}(k,t-1)+p(k-1)\mathbf{P}(k-1,t-1)$$
$$+(1-p)\delta_{k,1}, \qquad (5)$$

since

$$\sum_{i=1}^{t}\Pi_{i-1}(t-1)(1-\delta_{i,1})=1,$$

$$\sum_{i=1}^{t}\mathcal{P}_i(k,t-1)=\sum_{i=1}^{t-1}\mathcal{P}_i(k,t-1)=(t-1)\mathbf{P}(k,t-1).$$

The stationary behavior of $\mathbf{P}(k,t)$, mentioned in the previous section, can be checked from Eq. (5). Replacing the stationary quantity $\mathbf{P}(k)$ into Eq. (5), one gets

$$\mathbf{P}(k)=-pk\mathbf{P}(k)+p(k-1)\mathbf{P}(k-1)+(1-p)\delta_{k,1}, \qquad (6)$$

which can be solved for $\mathbf{P}(k)$:

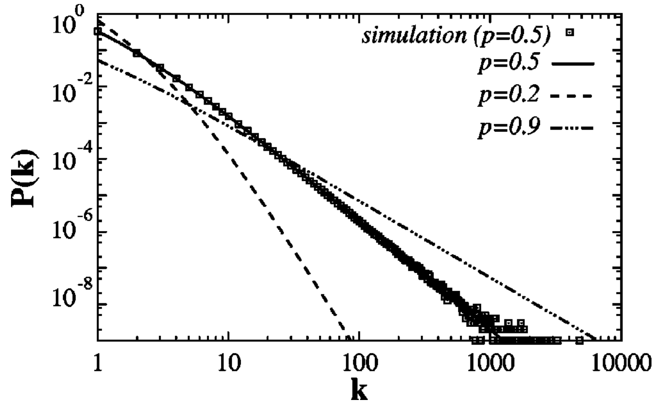

FIG. 2. Group size distribution in the asymptotic limit, for different p values.

$$\mathbf{P}(k)=\frac{\Gamma(k)\Gamma\left(2+\frac{1}{p}\right)}{\Gamma\left(k+1+\frac{1}{p}\right)}\frac{1-p}{1+p}\underset{k\to\infty}{\sim}k^{-1-1/p}. \qquad (7)$$

The stationary distribution of group size has a power-law decay with an exponent, $\gamma=1+1/p$, depending on the parameter p; see Fig. 2.

B. Analytic solution for the individual group size distribution

In the model the first group has an accentuated role since it always has at least one element because of the initial conditions. Therefore, Eq. (1) for the first group ($i=1$) has the following simpler form:

$$\mathcal{P}_1(k,t)=\mathcal{P}_1(k,t-1)-\frac{p}{t-1}k\,\mathcal{P}_1(k,t-1)$$
$$+\frac{p}{t-1}(k-1)\mathcal{P}_1(k-1,t-1). \qquad (8)$$

For $k=1$ in the above equation on the right-hand side the last term vanishes so the probability $\mathcal{P}_1(1,t)$ can be calculated easily:

$$\mathcal{P}_1(1,t)=\frac{\Gamma(t-p)}{\Gamma(t)\Gamma(1-p)}. \qquad (9)$$

For $k>1$ one can prove (see Appendix A) that the following equality holds:

$$\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t)=\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}. \qquad (10)$$

The analytic form of $\mathcal{P}_1(k,t)$ can be received from Eq. (10) by multiplying both sides with $(-1)^{l-1}\binom{k-1}{l-1}$ and summing up for $l=1\cdots k$,

$$\mathcal{P}_1(k,t)=\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}. \qquad (11)$$

In the case of $i>1$ we have to look at the whole Master Eq. (1). In this case the equality (10) does not hold because of the last factor in Eq. (1). Our assumption is that the probability $\mathcal{P}_i(k,t)$ will have a modified form:

$$\mathcal{P}_i(k,t)=\overbrace{\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}}^{\mathcal{P}_1(k,t)}\left[\sum_{b=i}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}(1-p)^{i-1}\right]. \tag{12}$$

The validity of the above form can be checked by replacing it back in Eq. (1); see Appendix B.

C. Mean value of group sizes

Replacing the analytic formula (12) into Eq. (4) one gets

$$\langle k_i\rangle,(t)=\sum_{k=1}^{t-i+1}k\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}\times\left[\sum_{b=i}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}(1-p)^{i-1}\right]. \tag{13}$$

The two sums can be transposed, $(\Sigma_{k=1}^{t-i+1}\Sigma_{l=1}^{k}=\Sigma_{l=1}^{t-i+1}\Sigma_{k=l}^{t-i+1})$ and

$$\sum_{k=l}^{t-i+1}k\binom{k-1}{l-1}=l\binom{t-i+2}{l+1},$$

so the mean value will have the following form:

$$\langle k_i\rangle(t)=\sum_{l=1}^{t-i+1}(-1)^{l-1}l\binom{t-i+2}{l+1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}\times\sum_{b=i}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}(1-p)^{i-1}. \tag{14}$$

D. Time dependent solution for the group size distribution $\mathbf{P}(k,t)$

In Sec. III A we calculated the stationary group size distribution directly from the master equation. Now we are interested in its dynamics. In order to compute that, we start from the definition (3) of $\mathbf{P}(k,t)$, and replace the solution we got for $\mathcal{P}_i(k,t)$ [Eq. (12)]:

$$\mathbf{P}(k,t)=\frac{1}{t}\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}\times\left[1+(1-p)\sum_{i=2}^{t}\sum_{b=i}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\times\binom{b-2}{i-2}p^{b-i}(1-p)^{i-2}\right]. \tag{15}$$

Transposing the two sums $\Sigma_{i=2}^{t}\Sigma_{b=i}^{t}=\Sigma_{b=2}^{t}\Sigma_{i=2}^{b}$, and taking into account that

$$\sum_{b=2}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}=\frac{\Gamma(1-lp)}{(1+lp)}\frac{\Gamma(t+1)}{\Gamma(t-lp)}-\frac{1}{1+lp},$$

one finally arrives at the time dependent distribution:

$$\mathbf{P}(k,t)=\underbrace{\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\left[\frac{1-p}{1+lp}\right.}_{\mathbf{P}(k,\infty)}\left.+\frac{p+lp}{1+lp}\frac{\Gamma(t-lp)}{\Gamma(t+1)\Gamma(1-lp)}\right]. \tag{16}$$

In the long time limit we will get back our result (7) since the second term in $[\cdots]$ decays for large t values with t^{-1-lp}, and the sum transforms into

$$\mathbf{P}(k,\infty)=\frac{1-p}{1+p}\frac{\Gamma(k)\Gamma(2+1/p)}{\Gamma(k+1+1/p)}.$$

IV. ASYMPTOTIC CASES

We study the $t\to\infty$ limit of $\mathcal{P}_i(k,t)$ and $\langle k_i\rangle(t)$. In the analytic formula for $\mathcal{P}_i(k,t)$, see Eq. (12), there are two components, $A(l,p,t)$ and $B(i,l,p,t)$, that depend on the time,

$$\mathcal{P}_i(k,t)=(1-p)^{i-1}\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\underbrace{\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}}_{A(l,p,t)}\underbrace{\left[\sum_{b=i}^{t}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}\right]}_{B(i,l,p,t)}. \tag{17}$$

The limit of the first term, $A(l,p,t)$, can be easily calculated,

$$\lim_{t\to\infty} A(l,p,t)=\frac{1}{\Gamma(1-lp)}t^{-lp}.$$

The second term in the long time limit $t \gg i,l$ will converge to a hypergeometric sum [14]:

$$\lim_{t\to\infty} B(i,l,p,t)=\tilde{B}(i,l,p)$$
$$=\frac{\Gamma(i)\Gamma(1-lp)}{\Gamma(i-lp)}{}_2F_1(i,i-1;i-lp;p).$$

For large time values the only time dependent term in Eq. (17) will be t^{-lp} which in case of large t is a fast decaying function of l. So in the case of $t \gg k$ we can assume that only the first term of the sum gives non-negligible component for $\mathcal{P}_i(k,t)$, i.e., distributions belonging to different k values will decay with the same exponent p; see Fig. 3,

$$\lim_{t\to\infty} \mathcal{P}_i(k,t)=t^{-p}(1-p)^{i-1}\frac{\Gamma(i)}{\Gamma(i-p)}{}_2F_1(i,i-1;i-p;p)+\mathcal{O}(t^{-2p}). \tag{18}$$

For large i values the above formula simplifies further, because in that case $\lim_{i\to\infty}{}_2F_1(i,i-1;i-p;p)\sim(1-p)^{1-i}$, and $\lim_{i\to\infty}[\Gamma(i)/\Gamma(i-p)]=i^p$:

$$\lim_{t,i\to\infty} \mathcal{P}_i(k,t)=\left(\frac{i}{t}\right)^p. \tag{19}$$

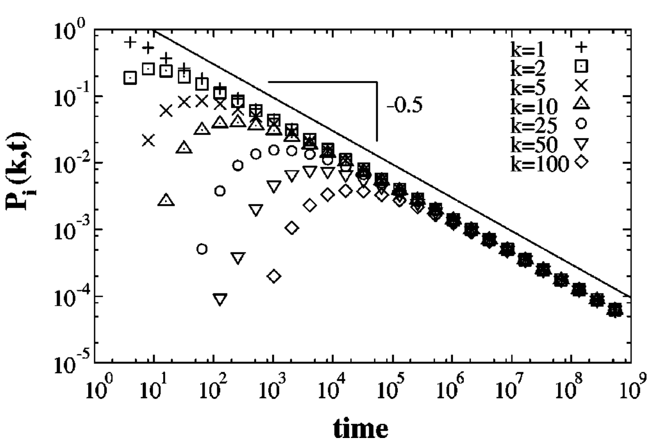

FIG. 3. Asymptotic behavior of $\mathcal{P}_i(k,t)$. We chose the parameters for $p=0.5$ and $i=2$. The figure demonstrates that in the long time limit the distributions for different k values converge to t^{-p}.

To study the asymptotic behavior of $\langle k_i\rangle(t)$ we start from the fact, that for small k values, $k \ll t$, the individual group size distribution, $\mathcal{P}_i(k,t)$, can be described by the first term of the sum, see Eq. (18), and for larger values $k \gtrsim t$ it has a fast decay; Fig. 4. A cutoff parameter k^* can be defined and we can assume that Eq. (4) transforms into

$$\langle k_i\rangle(t)\approx\sum_{k=1}^{k^*}k\,\mathcal{P}_i(k,t)=\mathcal{P}_i(1,t)\frac{k^*(k^*+1)}{2}. \tag{20}$$

The definition of k^* can be done in many ways. We defined k^* as the inflection point of $\mathcal{P}_i(k,t)$, hence

$$k^*=t^p\frac{\tilde{B}(i,3,p)}{\tilde{B}(i,4,p)}\frac{\Gamma(1-4p)}{\Gamma(1-3p)}+2+\mathcal{O}(t^{-p})$$
$$=t^p\frac{\Gamma(i-4p)}{\Gamma(i-3p)}\frac{{}_2F_1(i,i-1;i-3p;p)}{{}_2F_1(i,i-1;i-4p;p)}+2+\mathcal{O}(t^{-p}). \tag{21}$$

Replacing k^* into Eq. (20),

$$\langle k_i\rangle(t)\simeq t^p(1-p)^{i-1}\frac{\Gamma(i)}{\Gamma(i-p)}{}_2F_1(i,i-1;i-p;p)\times\left[\frac{\Gamma(i-4p)}{\Gamma(i-3p)}\frac{{}_2F_1(i,i-1;i-3p;p)}{{}_2F_1(i,i-1;i-4p;p)}\right]^2. \tag{22}$$

For large i values the above formula gets a simpler form, because in this case $\lim_{i\to\infty}[\Gamma(i)/\Gamma(i-p)]=i^p$,

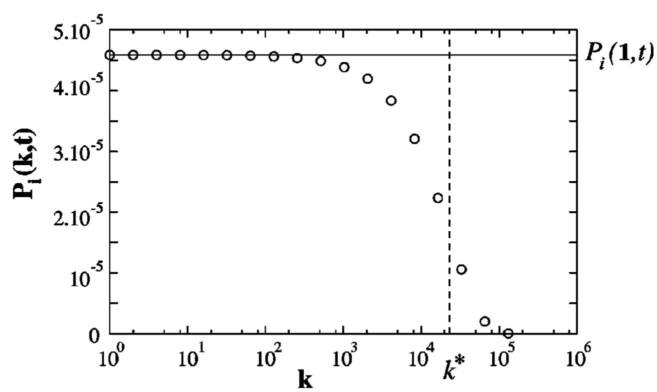

FIG. 4. Distribution of individual group size in the long time limit ($t=10^9$) as a function of the group size.

$\lim_{i\to\infty}[\Gamma(i-4p)/\Gamma(i-3p)]=i^{-p}$, $\lim_{i\to\infty}{}_2F_1(i,i-1;i-3p;p)=\lim_{i\to\infty}{}_2F_1(i,i-1;i-4p;p)$, $\lim_{i\to\infty}{}_2F_1(i,i-1;i-p;p)\sim(1-p)^{1-i}$,

$$\langle k_i\rangle(t)\underset{i\to\infty}{\approx}\left(\frac{t}{i}\right)^p. \tag{23}$$

V. DISCUSSION

In this paper we presented a simple preferential growth model consisting of a system of clusters with different sizes. We gave exact solutions for the main characteristic quantities as the distribution $\mathcal{P}_i(k,t)$ and the mean value $\langle k_i\rangle(t)$ of the individual group size as well as for the distribution of the average group size $\mathbf{P}(k,t)$.

The question rises why are such time dependent quantities of interest since most of the asymptotic scaling behavior can be obtained with much less labor. In fact, the growth models and network usually provide only a background for some dynamic process—an aspect which has not yet been paid enough attention to. If there is a strong separation of time scales, i.e., the growth is much smaller than the process itself, then it is satisfactory to concentrate on the asymptotics only. This is probably the case with the Internet or the WWW. However, in some cases such a separation of scales could be approximate only or even missing and then the importance of the full time dependence becomes apparent. We expect that in certain economic processes this will be the case.

An important aspect of the asymptotic scaling is universality. Similarly to other preferential growth models, our system exhibits nonuniversal parameter dependent scaling: the exponents depend on the parameter q (the probability of creating a new group). It is worth mentioning that the examples quoted in the introduction also show a wide variety of scaling exponents. Further interesting study would be to analyze a model where this parameter q depends on the time of the growth.

The system presented is not a network; the different groups are not linked to each other. However, for a specific value of the parameter, $p=0.5$, it can be interpreted as a kind of mean field network model. The clusters then denote the different nodes, and the particles are the links. The value $p=0.5$ means that in average in every second time step one new group and two elements are created (in the odd time steps the new element joins to an old group and in even time steps it will create a new group). The new group is the new node while the two new elements are the two ends of the new link, one is pointing to the old node, the other is to the new one. This case corresponds to the Barabasi's network model with parameter $m=1$ which means that the new node connects to *one* old sites. For this particular parameter choice our results agree with them got for the Barabasi's network model: $\mathbf{P}(k)\sim k^{-3}$, see Eq. (7), and $\langle k_i\rangle(t)\sim\sqrt{t/i}$, see Eq. (23).

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APPENDIX A:

We prove the assumption (10). If one multiplies Eq. (8) by $(-1)^{k-1}\binom{l-1}{k-1}$ and sums it up for $k=1\cdots l$ one gets

$$\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t)=\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t-1)$$
$$-\frac{p}{t-1}\Bigg[\underbrace{\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}k\,\mathcal{P}_1(k,t-1)}_{x}-\underbrace{\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}(k-1)\mathcal{P}_1(k-1,t-1)}_{y}\Bigg], \tag{A1}$$

where in the first term (x) we detach the last term of the sum:

$$x=\sum_{k=1}^{l-1}(-1)^{k-1}k\binom{l-1}{k-1}\mathcal{P}_1(k,t-1)+(-1)^{l-1}l\mathcal{P}_1(l,t-1). \tag{A2}$$

Taking into account that $\binom{l-1}{k}=[(l-k)/k]\binom{l-1}{k-1}$ the second term (y) can be rewritten as

$$y=\sum_{k=2}^{l}(-1)^{k-1}\binom{l-1}{k-1}(k-1)\mathcal{P}_1(k-1,t-1)$$
$$=-\sum_{k=1}^{l-1}(-1)^{k-1}(l-k)\binom{l-1}{k-1}\mathcal{P}_1(k,t-1), \tag{A3}$$

$$x-y=l\sum_{k=1}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t-1). \tag{A4}$$

Replacing the difference $x-y$ back to Eq. (A1) one gets the time evolution of the sum:

$$\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t)=\frac{t-1-lp}{t-1}\sum_{k=1}^{l}(-1)^{k-1}\binom{l-1}{k-1}\mathcal{P}_1(k,t-1), \tag{A5}$$

which leads back to our assumption (10).

APPENDIX B:

We prove the formula (12) for $\mathcal{P}_i(k,t)$ in the case of $i>1$, by replacing it into Eq. (1). The left-hand side of the equation after detaching the last term $(b=t)$ of the sum becomes

$$\mathcal{P}_i(k,t)=\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\binom{t-2}{i-2}p^{t-i}+\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)}\times\sum_{b=i}^{t-1}\frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}. \tag{B1}$$

The first term of the right-hand side becomes

$$\frac{t-1-kp}{t-1}\mathcal{P}_i(k,t-1)=(-1)^{k-1}\frac{\Gamma(t-kp)}{\Gamma(t)}\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-kp)}\binom{b-2}{i-2}p^{b-i}+\frac{t-1-kp}{t-1}\sum_{l=1}^{k-1}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-1-lp)}{\Gamma(t-1)}\times\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}, \tag{B2}$$

Taking into account that $(k-1)\binom{k-2}{l-1}=(k-l)\binom{k-1}{l-1}$, the second term will be

$$\frac{(k-1)p}{t-1}\mathcal{P}_i(k-1,t-1)=\frac{p}{t-1}\sum_{l=1}^{k-1}(-1)^{l-1}(k-l)\binom{k-1}{l-1}\frac{\Gamma(t-1-lp)}{\Gamma(t-1)}\times\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-lp)}\binom{b-2}{i-2}p^{b-i}. \tag{B3}$$

The sum of Eqs. (B2) and (B3) will be

$$\frac{t-1-kp}{t-1}\mathcal{P}_i(k,t-1)+\frac{(k-1)p}{t-1}\mathcal{P}_i(k-1,t-1)=(-1)^{k-1}\frac{\Gamma(t-kp)}{\Gamma(t)}\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-kp)}\binom{b-2}{i-2}p^{b-i}+\sum_{l=1}^{k-1}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)}\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-lp)}\frac{(b-2)}{(i-2)}p^{b-i}=\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}\frac{\Gamma(t-lp)}{\Gamma(t)}\sum_{b=i}^{t-1}\frac{\Gamma(b)}{\Gamma(b-lp)}\frac{(b-2)}{(i-2)}p^{b-i}, \tag{B4}$$

which will be equal to the second term of Eq. (B1). Simplifying with this term we get the remaining equation:

$$\binom{t-2}{i-2}p^{t-i}\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1}=\binom{t-2}{i-2}p^{t-i}\delta_{k,1}, \tag{B5}$$

which is true, because the sum equals $\delta_{k,1}$.

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